Local Learning Algorithms
for Transductive Classification, Clustering and Data Projection

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3. A Local Learning Approach for Clustering
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5. Summary
Supervised learning problem, training data $\{(x_1, y_1), \ldots, (x_n, y_n)\}$

**Global Learning Algorithms**
- A model $\mathcal{M}$ is built with all the training data $\{(x_i, y_i)\}_{i=1}^n$.
- $\mathcal{M}$ is used to predict the labels of any unseen test data.

**Local Learning Algorithms**
- For a given test point $x$, build a model $\mathcal{M}_x$ only using $\{(x_i, y_i)\}_{x_i \in N_x}$.
- Different models may be used for different test points.
Supervised learning problem, training data $(x_1, y_1), \ldots, (x_n, y_n)$

- **Global Learning Algorithms**
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- **Local Learning Algorithms**
  - For a given test point $x$, build a model $\mathcal{M}_x$ only using $\{(x_i, y_i)\}_{x_i \in N_x}$.
  - Different models may be used for different test points.
Local learning algorithms often outperform global ones since local models are trained only with the points that are related to the particular test data.

The good performance of local learning methods indicates:
The label of a point can be well estimated based on its neighbors.
Transductive Classification via Local Learning Regularization

A Local Learning Approach for Clustering

Local Learning Projections

Summary

Experimental Results

Remarks and Questions
Supervised and Transductive Classification

- **Binary Supervised Classification**
  - Given: \( \{(x_i, y_i)\}_{i=1}^l, x_i \in \mathcal{X} \subseteq \mathcal{R}^d, y_i \in \{-1, 1\} \).
  - Goal: Classification function \( f(x) \).
  - Learning only from labeled data.

- **Binary Transductive Classification (TC)**
  - Given:
    - Labeled data: \( \{(x_i, y_i)\}_{i=1}^l, x_i \in \mathcal{X} \subseteq \mathcal{R}^d, y_i \in \{-1, 1\} \).
    - Unlabeled data: \( \{(x_i)\}_{i=l+1}^{l+u}, \text{typically } u \gg l \).
  - Goal: Predict the class labels of the given unlabeled points.
  - Learning from both labeled and unlabeled data.
Supervised and Transductive Classification

- **Binary Supervised Classification**
  - Given: \( \{(x_i, y_i)\}_{i=1}^l, x_i \in \mathcal{X} \subseteq \mathbb{R}^d, y_i \in \{-1, 1\}. \)
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- **Binary Transductive Classification (TC)**
  - Given:
    - Labeled data: \( \{(x_i, y_i)\}_{i=1}^l, x_i \in \mathcal{X} \subseteq \mathbb{R}^d, y_i \in \{-1, 1\}. \)
    - Unlabeled data: \( \{(x_i)\}_{i=l+1}^{l+u}, \) typically \( u >> l. \)
  - Goal: Predict the class labels of the given unlabeled points.
  - Learning from both labeled and unlabeled data.
A Toy Example
Can We Ignore the Unlabeled Data?
Motivation

- Labeled data are expensive or difficult to obtain.
- Unlabeled data are much easier to get.
- Example: Web page classification.
Prior Assumption Is Important

- In many TC algorithms [Zhu et al., 2003, Belkin et al., 2005], each $x_i$ is assigned a real value $f_i$

$$y_i = \text{sign}(f_i) \quad l + 1 \leq i \leq l + u$$

- Main part: computing $f_i$ of each $x_i$.

- Key: the prior assumption about the properties that $f_i$ should have over the data points.

- Cluster assumption: If two data points $x_i$ and $x_j$ are on the same cluster, then the values of $f_i$ and $f_j$ should be similar to each other.
A typical formulation for TC [Zhu et al., 2003, Zhou et al., 2004]

\[
\min_{f \in \mathbb{R}^n} f^T R f + (f - y)^T C (f - y)
\]

where

- \( f = [f_1, \ldots, f_n]^T \in \mathbb{R}^n \).
- \( R \in \mathbb{R}^{n \times n} \): regularization matrix.
- \( y = [y_1, \ldots, y_l, 0, \ldots, 0]^T \in \mathbb{R}^n \).
- \( C \): a diagonal matrix. \( c_i = C_l > 0 \) for \( 1 \leq i \leq l \), and \( c_i = C_u \geq 0 \) for \( l + 1 \leq i \leq n \).
Laplacian Regularizer

- Graph Laplacian [Zhu et al., 2003], $R = L$.

$$f^T L f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} (f_i - f_j)^2$$

- $w_{ij}$, similarity between $x_i$ and $x_j$

$$w_{ij} = \exp(-\gamma \|x_i - x_j\|^2)$$

- Smoothness constraint: The labels should be similar among nearby points.
Local Learning Problem (LL-Problem)

For a data point $x_i$, given the values of $f_j$ at $x_j \in \mathcal{N}_i$, what should be the proper value of $f_i$ at $x_i$?

- $\{(x_j, f_j)\}_{x_j \in \mathcal{N}_i} \rightarrow f_i$, a learning problem.
- Local Learning Regularizer:

$$\sum_{i=1}^{n} (f_i - o_i(x_i))^2$$

- $o_i(\cdot)$, trained with $\{(x_j, f_j)\}_{x_j \in \mathcal{N}_i}$
- Idea: $f_i$ should be well estimated locally based on the neighboring points of $x_i$, using supervised learning algorithms.
TC via Local Learning

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- Idea: $f_i$ should be well estimated locally based on the neighboring points of $\mathbf{x}_i$, using supervised learning algorithms.
Computing $o_i(x_i)$  

Following [Bottou & Vapnik, 1992],

- Linear model

$$o_i(x) = w_i^T (x - x_i) + b_i, \quad \forall x \in \mathbb{R}^d$$

- Training data, $\{(x_j, f_j)\}_{x_j \in \mathcal{N}_i}$. Training problem:

$$\min_{w_i \in \mathbb{R}^d, b_i \in \mathbb{R}} \lambda \|w_i\|^2 + \sum_{x_j \in \mathcal{N}_i} (o_i(x_j) - f_j)^2$$

- Solution

$$o_i(x_i) = \alpha_i^T f_i$$

$f_i \in \mathbb{R}^{n_i}$, the vector $[f_j]^T$ for $x_j \in \mathcal{N}_i$.

$o_i(x_i)$ can be computed analytically, even if we do not know the values of $\{f_j\}_{x_j \in \mathcal{N}_i}$. 
Computing $o_i(x_i)$ (2/2)

Matrix form of $o_i(x_i)$

$$\mathbf{o} = \mathbf{A}\mathbf{f}$$

$$\mathbf{o} = [o_1(x_1), \ldots, o_n(x_n)]^\top, \mathbf{f} = [f_1, \ldots, f_n]^\top$$

An example:

$$o_1(x_1) = a \times f_2 + b \times f_3, \quad o_2(x_2) = c \times f_1 + d \times f_4, \ldots.$$  

then

$$
\begin{pmatrix}
  o_1(x_1) \\
  o_2(x_2) \\
  \vdots
\end{pmatrix} =
\begin{pmatrix}
  0 & a & b & 0 & \cdots \\
  c & 0 & 0 & d & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots
\end{pmatrix}
$$
Local Learning Regularizer:

- Local Learning Regularizer:
  \[
  \sum_{i=1}^{n} (f_i - o_i(x_i))^2 = \|f - o\|^2 = \|f - Af\|^2 = f^\top R_L f
  \]

- Quadratic objective:
  \[
  \min_{f \in \mathbb{R}^n} f^\top R_L f + (f - y)^\top C(f - y)
  \]
Comparison with Laplacian Regularizer

- Laplacian regularizer:
  \[
  f^T L f = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} w_{ij} (f_i - f_j)^2
  \]

- Current explanations: smoothness constraints, manifold regularization, random walk.

- Implicit answer to LL-problem: Setting \( \frac{\partial}{\partial f} f^T L f \) to 0, we have,
  \[
  f_i = \frac{\sum_{x_j \in \mathcal{N}_i} w_{ij} f_j}{\sum_{x_j \in \mathcal{N}_i} w_{ij}}
  \]
Leave-One-Out (LOO) Error

- Quadratic objective

\[
\min_{f \in \mathbb{R}^n} f^\top R f + (f - y)^\top C (f - y)
\]

Solution: \( f = (R + C)^{-1} Cy \), let \( M = (R + C)^{-1} \).

- LOO procedure for TC: In the \( i \)-th iteration (\( 1 \leq i \leq l \)), \((x_i, y_i) \rightarrow x_i\), solution \( f^{(i)} \).

- To compute LOO error: We only need to know \( f_i^{(i)} \).

### Computing LOO Error Efficiently

\[
f_i^{(i)} = \frac{f_i - C_i y_i m_{ii}}{1 - (C_i - C_u) m_{ii}} \quad 1 \leq i \leq l
\]
## Experimental Results

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Lap-Reg</th>
<th>NLap-Reg</th>
<th>LLE-Reg</th>
<th>LL-Reg</th>
</tr>
</thead>
<tbody>
<tr>
<td>g241c</td>
<td>39.00 ± 2.23</td>
<td>45.00 ± 3.92</td>
<td>41.46 ± 4.51</td>
<td>21.36 ± 3.67</td>
</tr>
<tr>
<td>g241d</td>
<td>36.12 ± 1.50</td>
<td>43.31 ± 3.30</td>
<td>40.15 ± 4.05</td>
<td>22.51 ± 1.79</td>
</tr>
<tr>
<td>Digit1</td>
<td>3.02 ± 0.84</td>
<td>2.91 ± 0.59</td>
<td>2.54 ± 0.72</td>
<td>2.63 ± 0.66</td>
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<tr>
<td>USPS</td>
<td>7.09 ± 3.37</td>
<td>4.60 ± 2.04</td>
<td>4.70 ± 1.86</td>
<td>3.67 ± 1.24</td>
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<tr>
<td>COIL</td>
<td>11.11 ± 2.72</td>
<td>10.71 ± 3.29</td>
<td>13.61 ± 4.01</td>
<td>12.04 ± 2.30</td>
</tr>
<tr>
<td>BCI</td>
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<td>31.15 ± 5.02</td>
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<td>Text</td>
<td>29.29 ± 2.36</td>
<td>23.55 ± 3.55</td>
<td>50.11 ± 0.37</td>
<td>24.23 ± 3.28</td>
</tr>
<tr>
<td>Banana</td>
<td>14.26 ± 1.69</td>
<td>14.15 ± 1.96</td>
<td>17.09 ± 2.48</td>
<td>12.75 ± 1.70</td>
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<tr>
<td>Diabetis</td>
<td>32.80 ± 2.54</td>
<td>31.93 ± 2.71</td>
<td>33.31 ± 2.51</td>
<td>27.63 ± 2.40</td>
</tr>
<tr>
<td>German</td>
<td>31.76 ± 2.51</td>
<td>30.82 ± 1.40</td>
<td>33.69 ± 2.95</td>
<td>29.95 ± 2.49</td>
</tr>
<tr>
<td>Image</td>
<td>14.39 ± 1.63</td>
<td>14.28 ± 1.88</td>
<td>18.67 ± 2.44</td>
<td>12.08 ± 2.07</td>
</tr>
<tr>
<td>Ringnorm</td>
<td>19.06 ± 2.17</td>
<td>9.66 ± 0.86</td>
<td>11.85 ± 1.48</td>
<td>10.28 ± 0.38</td>
</tr>
<tr>
<td>Splice</td>
<td>37.48 ± 3.75</td>
<td>36.01 ± 8.50</td>
<td>39.36 ± 2.31</td>
<td>27.27 ± 5.05</td>
</tr>
<tr>
<td>Twonorm</td>
<td>4.17 ± 1.30</td>
<td>4.05 ± 1.29</td>
<td>7.54 ± 1.33</td>
<td>3.35 ± 1.02</td>
</tr>
<tr>
<td>Waveform</td>
<td>17.09 ± 3.27</td>
<td>16.88 ± 3.28</td>
<td>19.02 ± 1.80</td>
<td>13.50 ± 1.94</td>
</tr>
</tbody>
</table>
Remarks

- Local learning regularization for TC.
- A flexible framework, adapting various learning algorithms for TC.
- Examining some current regularizers under this framework.
- An efficient way to compute the LOO error.

Questions

- No labeled points at all, unsupervised learning?
- Nonlinear local models?
Remarks

- Local learning regularization for TC.
- A flexible framework, adapting various learning algorithms for TC.
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Questions

- No labeled points at all, unsupervised learning?
- Nonlinear local models?
A Local Learning Approach for Clustering
The clustering problem

Clustering Problem

- $n$ data points, $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x}_i \in \mathcal{X} \subseteq \mathbb{R}^d$.
- $c > 0$.
- goal: partition the given data into $c$ clusters.
- current methods: k-means, single-link, spectral clustering.
Representation of Clustering Results

- Partition Matrix: \( P \in \{0, 1\}^{n \times c} \)
- Scaled Partition Matrix: \( F = P(P^T P)^{-\frac{1}{2}} \)

\[
P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}
\]

\[
F = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \sqrt{3} \\ 0 & 1 \sqrt{3} \\ 0 & 1 \sqrt{3} \end{pmatrix}
\]

- Two useful properties:

\[
F^T F = (P^T P)^{-\frac{1}{2}} P^T P (P^T P)^{-\frac{1}{2}} = I
\]

\[
P = P(F) = \text{Diag}(FF^T)^{-\frac{1}{2}} F
\]
Basic Idea

**LL-Problem, $F \in \mathbb{R}^{n \times c}, f^l_i$**

For a data point $x_i$ and a cluster $C_l$, given the values of $f^l_j$ at $x_j \in \mathcal{N}_i$, what should be the proper value of $f^l_i$ at $x_i$?

**Objective function:**

$$
\min_{F \in \mathbb{R}^{n \times c}} \sum_{l=1}^{c} \sum_{i=1}^{n} (f^l_i - o^l_i(x_i))^2 \quad (1)
$$

s.t. $F$ is a scaled partition matrix

$$
o^l_i(\cdot)$, trained with $\{(x_j, f^l_j)\}_{x_j \in \mathcal{N}_i}$.

Mingrui Wu  Local Learning Algorithms
**Basic Idea**

**LL-Problem, $\mathbf{F} \in \mathbb{R}^{n \times c}, f^l_i$**

For a data point $\mathbf{x}_i$ and a cluster $C_l$, given the values of $f^l_j$ at $\mathbf{x}_j \in \mathcal{N}_i$, what should be the proper value of $f^l_i$ at $\mathbf{x}_i$?

**Objective function:**

$$\min_{\mathbf{F} \in \mathbb{R}^{n \times c}} \sum_{l=1}^{c} \sum_{i=1}^{n} (f^l_i - o^l_i(\mathbf{x}_i))^2 \quad (1)$$

s.t. \quad $\mathbf{F}$ is a scaled partition matrix \quad (2)

$$o^l_i(\cdot), \text{ trained with } \{(\mathbf{x}_j, f^l_j)\}_{\mathbf{x}_j \in \mathcal{N}_i}.$$
Computing $o_i^l(x_i)$

- Training data: $\{(x_j, f_j^l)\}_{x_j \in \mathcal{N}_i}$. Kernel learning algorithms:
  
  $o_i^l(x_i) = \sum_{x_j \in \mathcal{N}_i} \beta_{ij}^l K(x_i, x_j) = k_i^T \beta_i^l$

- Kernel ridge regression:
  
  $$\min_{\beta_i^l \in \mathbb{R}^{n_i}} \lambda (\beta_i^l)^T K_i \beta_i^l + \|K_i \beta_i^l - f_i^l\|^2$$

  $$K_i = [K(x_u, x_v)] \in \mathbb{R}^{n_i \times n_i}, \text{ for } x_u, x_v \in \mathcal{N}_i.$$

- Solution of kernel ridge regression: $\beta_i^l = (K_i + \lambda I)^{-1} f_i^l$

- $o_i^l(x_i) = k_i^T (K_i + \lambda I)^{-1} f_i^l = \alpha_i^T f_i^l,$
  
  $\alpha_i^T = k_i^T (K_i + \lambda I)^{-1}$
Computing \( o_i^l(\mathbf{x}_i) \)

- Training data: \( \{(\mathbf{x}_j, f_j^l)\}_{\mathbf{x}_j \in \mathcal{N}_i} \). Kernel learning algorithms:
  \[
o_i^l(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{N}_i} \beta_i^l K(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{k}_i^T \beta_i^l
  \]

- Kernel ridge regression:
  \[
  \min_{\beta_i^l \in \mathbb{R}^{n_i}} \lambda(\beta_i^l)^T K_i \beta_i^l + \left\| K_i \beta_i^l - f_i^l \right\|^2
  \]
  \[
  \mathbf{K}_i = [K(\mathbf{x}_u, \mathbf{x}_v)] \in \mathbb{R}^{n_i \times n_i}, \text{ for } \mathbf{x}_u, \mathbf{x}_v \in \mathcal{N}_i.
  \]

- Solution of kernel ridge regression: \( \beta_i^l = (K_i + \lambda \mathbf{I})^{-1} f_i^l \)

- \( o_i^l(\mathbf{x}_i) = \mathbf{k}_i^T (K_i + \lambda \mathbf{I})^{-1} f_i^l = \mathbf{\alpha}_i^T f_i^l \),
  \[
  \mathbf{\alpha}_i^T = \mathbf{k}_i^T (K_i + \lambda \mathbf{I})^{-1}
  \]
Quadratic Objective Function

- **Matrix form**:  \( \mathbf{o}^l = A\mathbf{f}^l \)
  
  \( \mathbf{o}^l = [o^l_1(x_1), \ldots, o^l_n(x_n)]^T, \mathbf{f}^l = [f^l_1, \ldots, f^l_n]^T \)

- **Objective function**:  \[
  \min_{\mathbf{F} \in \mathbb{R}^{n \times c}} \sum_{l=1}^c \sum_{i=1}^n (f^l_i - o^l_i(x_i))^2 = \sum_{l=1}^c \|\mathbf{f}^l - \mathbf{o}^l\|^2 \quad (3)
  \]

  s.t.  \( \mathbf{F} \) is a scaled partition matrix  \( (4) \)

- **Quadratic objective function**:  \[
  \min_{\mathbf{F} \in \mathbb{R}^{n \times c}} \sum_{l=1}^c \|\mathbf{f}^l - A\mathbf{f}^l\|^2 = \sum_{l=1}^c (\mathbf{f}^l)^T \mathbf{T}\mathbf{f}^l = trace(\mathbf{F}^T \mathbf{T} \mathbf{F})
  \]

  s.t.  \( \mathbf{F} \) is a scaled partition matrix

  \( \mathbf{T} = (\mathbf{I} - \mathbf{A})^T(\mathbf{I} - \mathbf{A}) \)
Quadratic Objective Function

- **Matrix form:** $o^l = Af^l$
  
  $o^l = [o^l_1(x_1), \ldots, o^l_n(x_n)]^T$, $f^l = [f^l_1, \ldots, f^l_n]^T$

- **Objective function:**

  $$\min_{F \in \mathbb{R}^{n \times c}} \sum_{l=1}^{c} \sum_{i=1}^{n} (f^l_i - o^l_i(x_i))^2 = \sum_{l=1}^{c} \|f^l - o^l\|^2$$  \hspace{1cm} (3)

  s.t. $F$ is a scaled partition matrix \hspace{1cm} (4)

- **Quadratic objective function:**

  $$\min_{F \in \mathbb{R}^{n \times c}} \sum_{l=1}^{c} \|f^l - Af^l\|^2 = \sum_{l=1}^{c} (f^l)^T T f^l = \text{trace}(F^T TF)$$

  s.t. $F$ is a scaled partition matrix

  $$T = (I - A)^T (I - A)$$
Relaxation

Relax $F$ to continuous domain

$$\min_{F \in \mathbb{R}^{n \times c}} \quad trace(F^TTF)$$

s.t. $F^TF = I$

Solution:

$$\{ F^*R : R \in \mathbb{R}^{c \times c}, \quad R^TR = I \}$$

where $F^* \in \mathbb{R}^{n \times c}$, consists of the $c$ smallest eigenvectors of $T$
Discretization: Obtaining the Final Clustering Result

- Get real valued partition matrix: $P^* = P(F^*)$

- A property: $P(F^*R) = P^*R \quad \forall R^\top R = I$.

- $F^*R$ close to the true SPM, $P^*R$ close to the corresponding discrete PM.

- Compute $R$ and discrete PM $P$ [Yu & Shi, 2003]:

\[
\begin{align*}
\min_{P \in \mathbb{R}^{n \times c}, R \in \mathbb{R}^{c \times c}} & \quad \|P - P^*R\|^2 \\
\text{subject to} & \quad P \in \{0, 1\}^{n \times c}, \quad P1_c = 1_n \\
& \quad R^\top R = I
\end{align*}
\]
## Numerical Results

<table>
<thead>
<tr>
<th></th>
<th>U3568</th>
<th>U49</th>
<th>UMist</th>
<th>UMist5</th>
<th>News4a</th>
<th>News4b</th>
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<tbody>
<tr>
<td><strong>NMI, cosine</strong></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Spec-Clst</td>
<td>0.6575</td>
<td>0.3608</td>
<td>0.7483</td>
<td>0.8810</td>
<td>0.6468</td>
<td>0.5765</td>
</tr>
<tr>
<td>LLCA1</td>
<td>0.8720</td>
<td>0.6241</td>
<td>0.8003</td>
<td>1</td>
<td>0.7587</td>
<td>0.7125</td>
</tr>
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<td>LLCA2</td>
<td>0.8720</td>
<td>0.6241</td>
<td>0.7889</td>
<td>1</td>
<td>0.7587</td>
<td>0.7125</td>
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<td>k-means</td>
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<td><strong>Error (%), Gaussian</strong></td>
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</table>
Remarks

- Adapting the local learning idea for the clustering problem.
- Cost function: the cluster label of each data point can be well estimated based on its neighbors.
- Nonlinear local models.
- Easy implementation, encouraging results.

Questions

- The LL approach is effective to explore the relationship among neighboring points. Any other applications?
Remarks

- Adapting the local learning idea for the clustering problem.
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Questions

- The LL approach is effective to explore the relationship among neighboring points. Any other applications?
Local Learning Projections
Linear Projection

- Training data $(x_1, y_1), \ldots, (x_n, y_n), x_i \in \mathcal{X} \subset \mathbb{R}^d, y_i \in \{1, 2, \ldots, c\}$.

- Goal: Find a low dimensional subspace of $\mathcal{X}$, which retains the discriminating information for classification.

- Projection matrix $P \in \mathbb{R}^{d \times p}, p < d, x \rightarrow P^T x$.

- Reducing noise, removing redundant information irrelevant to the classification task.
Some Linear Projection Methods

- Principal Component Analysis (PCA)
- Linear Discriminant Analysis (LDA)
- Locality Preserving Projection (LPP) [He & Niyogi, 2004]
Local Learning Projections (LLP)

Idea

The projection of a point can be well estimated based on its neighbors in the same class.

Objective function:

\[
\min_{P \in \mathbb{R}^{d \times p}, F \in \mathbb{R}^{p \times n}} \sum_{l=1}^{p} \sum_{i=1}^{n} (f'_l - o'_i(x_i))^2 = \sum_{l=1}^{p} \|f'_l - o'_l\|^2
\]

s.t.

\[ F = P^T X \]

\[ P^T P = I \]

\[ X = [x_1, \ldots, x_n], \quad o'_i(\cdot), \text{ trained with } \{(x_j, f'_j)\}_{x_j \in \mathcal{N}_i} \]
Local Learning Projections (LLP)

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The projection of a point can be well estimated based on its neighbors in the same class.

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\[
\min_{\mathbf{P} \in \mathbb{R}^{d \times p}, \mathbf{F} \in \mathbb{R}^{p \times n}} \sum_{l=1}^{p} \sum_{i=1}^{n} (f_i^l - o_i^l(x_i))^2 = \sum_{l=1}^{p} \|f^l - o^l\|^2 \tag{8}
\]

s.t. \[\mathbf{F} = \mathbf{P}^\top \mathbf{X}\] \[\mathbf{P}^\top \mathbf{P} = \mathbf{I}\] \[\tag{9} \tag{10}\]

\[\mathbf{X} = [x_1, \ldots, x_n], \ o_i^l(\cdot), \text{trained with } \{(x_j, f_j^l)\}_{x_j \in \mathcal{N}_i}\]
Consider the case $p = 1$, $f_i = p^\top x_i$

- Basic idea of LPP: $E_{LPP}(f) = \frac{1}{2} \sum_{i,j} (f_i - f_j)^2 w_{ij}$

- Setting $\frac{\partial}{\partial f} E_{LPP}(f)$ to 0, optimal $f$: $f_i = \frac{\sum_{x_j \in N_i} w_{ij} f_j}{\sum_{x_j \in N_i} w_{ij}}$

- LLP explicitly requires that $f_i$ can be well estimated based on the neighbors of $x_i$, while LPP specifies this implicitly.

- In LLP, $f_i$ is estimated by $o_i(x_i)$, which is trained with well established regression approaches, while in LPP, $f_i$ is estimated with the local average.
Locality Preserving Projections (LPP)

Consider the case $p = 1, f_i = p^\top x_i$

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Comparison with PCA (1/2)

**Global estimation error:**
Input data \(\{x_i\}_{i=1}^n\), a vector \(f = [f_1, \ldots, f_n]^T \in \mathbb{R}^n\), define

\[
E_{global}(f) = \sum_{i=1}^{n} (f_i - o_{all}(x_i))^2
\]

\(o_{all}(\cdot)\): trained with \(\{(x_i, f_i)\}_{i=1}^n\), using kernel ridge regression.

**Proposition**

Let \(\bar{f} = [\bar{f}_1, \ldots, \bar{f}_n]^T \in \mathbb{R}^n\), where \(\bar{f}_i\) denotes the projection value of \(x_i\) given by KPCA algorithm. Then among all the unit length vectors, \(\bar{f}/||\bar{f}||\) is the one with the minimal global estimation error. Namely,

\[
\frac{\bar{f}}{||\bar{f}||} = \arg \min_{f^Tf = 1} E_{global}(f)
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\[
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\]
Comparison with PCA (2/2)

- PCA minimizes **global** estimation error:
  \[ E_{global}(\mathbf{f}) = \sum_{i=1}^{n} (f_i - o_{all}(\mathbf{x}_i))^2 \]

  \( o_{all}(\cdot) \): trained with \( \{(\mathbf{x}_i, f_i)\}_{i=1}^{n} \). Not using class labels.

- LLP minimizes **local** estimation error:
  \[ E_{local}(\mathbf{f}) = \sum_{i=1}^{n} (f_i - o_{i}(\mathbf{x}_i))^2 \]

  \( o_{i}(\cdot) \): trained with \( \{(\mathbf{x}_j, f_j)\}_{x_j \in N_i} \). Using class labels.
## Experimental Results

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<tr>
<th>Dataset</th>
<th>m</th>
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Adapting the local learning idea for data projection.
LLP can keep local relationship among neighboring points.
A new explanation of PCA, based on which we can see the advantages of LLP over PCA.
Summary

- Adapting the local learning idea for TC, clustering and data projection.
- Investigating existing methods from the local learning point of view.
- Asking the relevant question explicitly.

Future Works

- Other applications, image segmentation, image matting, ranking, etc.
- Multi-view clustering and transductive learning.
- Local learning on graphs.
Thank you very much for your attention!
On manifold regularization. 
*AISTATS05*

Local learning algorithms. 
*Neural Computation, 4*, 888–900.

Locality preserving projections. 
In S. Thrun, L. Saul and B. Schölkopf (Eds.), *Advances in neural information processing systems 16*.

Face recognition using laplacianfaces. 

Multiclass spectral clustering.