Introduction to Category Theory

Olivier Bousquet
Department of Empirical Inference
Max Planck Institute of Biological Cybernetics
olivier.bousquet@tuebingen.mpg.de

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Informal Description

- "Generalized mathematical theory of structures"
- Goal: "reveal the universal properties of structures via their relationships"
- Emphasis on relationships rather than on objects
- Uniform treatment of the notion of structure
- Provides new foundations to mathematics (debated)
- Many applications in mathematics, mathematical physics, computer science...

Definition

A category is given by

- Collection of objects
- For each pair of objects a, b, collection of morphisms (or arrows) $Mor(a, b)$
- Composition: $Mor(a, b) \times Mor(b, c) \longrightarrow Mor(a, c)$
- Identity: $id_a \in Mor(a, a)$

Axioms

- Associativity of composition: $f \circ (g \circ h) = (f \circ g) \circ h$
- Identity: $(id_a \circ f) = (f \circ id_b) = f$

Examples (I)

- Sets with functions
- Groups with group homomorphisms
- Topological spaces with continuous maps
- Vector spaces with linear maps
- Differentiable manifolds with smooth maps

$\longrightarrow$ Typical: objects are structured sets and morphisms are structure preserving maps

Examples (II)

But categories are more general than this:

- Preordered set: elements and comparisons

- Group: 1 object and morphisms are group elements

- Directed graph: objects are vertices and morphisms are paths

Universality (I)

Unification of mathematical structures: factorization of recurring constructions.

Product

- Given two objects a, b in C
- The product is the triple (c, p, q)
- c object of C (representing the "cartesian product $a \times b$")
- $p : c \longrightarrow a$ and $q : c \longrightarrow b$ morphisms (representing "projections")
- Universality property: for all $d \in C$ and $f : d \longrightarrow a$, $g : d \longrightarrow b$, $\exists!$ morphism $h : d \longrightarrow c$ s.t. $p \circ h = f$ and $q \circ h = g$

Examples

- Sets: cartesian product
- Groups: product of groups (with appropriate group structure)
- Preordered set: least upper bound
- Exercise: product in a graph ?

$\longrightarrow$ Because emphasis on morphisms, structures are automatically transported to products
$\longrightarrow$ Notion of smallest product included in the universality property

Diagrams

- Convenient way of representing statements

- Commutation means equality of paths

- Example: axioms of categories (associativity, unity), product

$\longrightarrow$ See whiteboard !

Relationships: Functors

Unification of mathematical structures: study of relationships between structures

- A functor is a *morphism* of categories (preserves structure)
- $F : C \longrightarrow D$ maps objects of C to objects of D and morphisms to morphisms
- $f : c \longrightarrow c'$ is mapped to $F(f) : F(c) \longrightarrow F(c')$
- $F(id_c) = id_{F(c)}$
- $F(f \circ g) = F(f) \circ F(g)$

Examples

- Power set functor $P : Set \longrightarrow Set$
  - $\star$ Maps X to $P(X)$ set of subsets of X
  - $\star$ Maps $f : X \rightarrow Y$ to $P(f) : S \subset X \mapsto f(S) \subset Y$
- Linear group of invertible matrices $GL_n : CRing \longrightarrow Grp$
- Homotopy group $Top \longrightarrow Grp$
- Forgetful $Grp \longrightarrow Set$

Relationships: Natural transformations

Relationships between functors $F, G : C \longrightarrow D$ (morphism of functors)

- For each $c \in C$, consider a morphism $h_c : F(c) \longrightarrow G(c)$
- Commutativity condition: $f : c \longrightarrow d$, $G(f) \circ h_c = h_d \circ F(f)$

$\longrightarrow$ Diagram on whiteboard

Examples

- Determinant (morphism from complex to real rings, F is GL_n and G is group of invertible elements)

- Identity to power set

Limits

- More general than products
- Given categories C, J (J is the index set) and functor $F : J \longrightarrow C$ (defines which objects are used in the "product")
- Define limit object $r \in C$
- For each $c \in C$, $j \in J$, a morphism $h_{cj} : c \longrightarrow F(j)$
- Universality property

$\longrightarrow$ Product are a special case with $J = \{1, 2\}$

Adjoint Functors

- More fundamental than limits, cornerstone of the theory
- Categories C, D and functors $F : C \longrightarrow D$, $G : D \longrightarrow C$
- In addition, a map $\phi : Ob(C) \times Ob(D) \longrightarrow$bijections of morphisms $Mor_D(F(c), d) \equiv Mor_C(c, G(d))$
- Commutativity property

Examples

- Set, Vct, F free vector space, G forgetful functor, $\phi(S, V)$ bijection between linear maps $F(S) \longrightarrow V$ and maps $S \longrightarrow G(V)$

- $C \times C, C$, F product functor, G diagonal functor ($c \mapsto (c, c)$)

- C^J, C, F limit functor, G diagonal functor

- $\{1\}, C$, F terminal object, G diagonal functor

Relationship with Set Theory

- In set theory, identify equal elements
- In CT, identify isomorphic elements
- In CT, classes of objects need not be sets
- Category of all sets is defined even if there is no set of all sets

Higher Dimensional

Climb the category theoretic ladder to investigate more relationships
Start with notion of cells (diagram)

- 0-cells: points
- 1-cells: arrows
- 2-cells: arrows between arrows
- 3-cells: ...

n-categories

- 0-categories: sets
- 1-categories: standard categories
- 2-categories: e.g. Cat with categories, functors and natural transformations
- More general example: topological spaces and paths, maps between paths...

Axioms

- Appropriate properties for cell relationships (iteratively using definition of categories)
- In 2-categories: e.g. horizontal and vertical composition with $(\alpha\alpha') \otimes (\beta\beta') = (\alpha \otimes \beta)(\alpha' \otimes \beta')$
- Replace equality by isomorphism: weak n-categories (more interesting) but need coherence relationships
- Unfortunately, combinatorial effects (which equalities to weaken ?) prevents from easily finding general definition

$\longrightarrow$ Major topic of research in foundations of mathematics, mathematical physics (topological quantum field theory) and philosophy